Timing Errors in a Chain of Regenerative Repeaters, II

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The behavior of the timing jitter in a long chain of repeaters is shown to depend on the spectral properties of a linear operator which maps the space of bounded sequences into itself. As the number of repeaters increases indefinitely, so does the mean value of the jitter. The variation about this mean value remains bounded only for certain highly constrained pulse trains (e.g., periodic, finite, etc.), but it is otherwise unbounded.

I. INTRODUCTION

We showed in a previous discussion that the pulse displacements at the output of a chain of repeaters may be represented by a linear transformation of the pulse displacements at the output of the first repeater.* The linear transformation turns out to be a simple function of a basic operator T which, in essence, represents the action of the repeater on the incoming jitter. Though the operator T depends directly on the manner in which the repeater extracts its timing information from the incoming pulse train, it is believed that there would be no basic difference in the major results obtained or in the method of analysis for different timing extractors. We have assumed that the timing information extractor is a tuned circuit with a finite but fairly high Q and the source of jitter is the mistuning in the tuned circuit. Other sources of jitter often lend themselves to a similar mode of investigation.

The rest of the discussion in Part I concerned the class of periodic pulse trains. The problem reduces, in such cases, to a consideration of linear transformations in a finite dimensional space. For a periodic pulse train with period m, it was shown that the variance of the jitter remains bounded for an indefinitely long string of repeaters.

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* We shall assume that the reader is familiar with the contents of Part I of this paper: B. K. Kinariwala, Timing Errors in a Chain of Regenerative Repeaters, I, this issue, pp. 1769–1780.

Unfortunately, the above results do not let us draw any conclusions for the behavior of the variance when the pulse train is not periodic, but infinite, in length. For example, if there existed a bound M, on the variance, which was not a function of m, then we can let the period become infinite and conclude that the variance was bounded for the indefinitely long random pulse train. However, it is not apparent whether M is dependent on m or not. The value of the variance is determined by the number of eigenvalues of the pertinent operator, their location, and the algebraic signs of the corresponding eigenvectors. It seems reasonable, therefore, that the bound on the variance is a function of the period m. The behavior of this function as m approaches infinity will determine whether the variance is bounded in the nonperiodic case. We do not pursue the matter in this direction because it is not easy to express the above function in a simple manner.

Instead, we investigate the general problem directly in the infinite dimensional space. We establish that the basic operator T maps the normed linear space l_p into l_p for $1 \leqq p \leqq \infty$. Next, we show that the domain of T for our problem is the space l_∞.* We determine the conditions under which the variance is bounded, and we conclude that there is no bound on the variance of the jitter for the random (infinite) pulse train. The conclusion remains valid for any specification of dependence or independence of the random variables $\mathfrak{a}_n$ which take on the value one if a pulse is present at time $t = (-n\tau)$, but they are zero otherwise. Even a bound on the maximum number of successive zeros in the pulse trains does not seem to alter our result. Only when the operator T is restricted to a finite dimensional space does the variance remain finite. Such a restriction occurs for finite pulse trains, periodic pulse trains, nonperiodic pulse trains which eventually take on a periodic behavior, and so on.

The organization of the paper is in the nature of a proof with digressions. Though these digressions are extraneous to the discussion of the boundedness of the variance, they do serve to bring out some interesting points. We begin with the mathematical statement of the problem, which includes certain modifications of the previous statement. Next, we examine the elementary operator T and its properties such as boundedness, domain, and spectrum. We then proceed to the discussion of whether the variance of the jitter is bounded or unbounded. We close with a brief discussion of the results.

* The space l_∞ is a normed linear space which is complete. Hence, it is a Banach space.

II. STATEMENT OF PROBLEM

The purpose of a restatement of the problem here is to make certain desirable modifications. We also refer to a possible alternate formulation which, except for an occasional observation, we shall not pursue.

We are interested in studying the behavior of the equation

$$Y = \lim_{l \to \infty} \left[\sum_{\nu=0}^{l} T^{\nu} \right] X, \tag{1}$$

where X and Y represent the input and output jitter vectors, respectively, for a long chain of repeaters. By input jitter we mean input to the second repeater in the chain, and it is understood that the input to the first repeater is a jitter-free pulse train. The linear operator T represents the action of the repeater on the incoming jitter, and we shall describe it presently. The simple form of (1) is obtained by assuming that the mistunings, which appear as coefficients in the power series in T, are identical. This assumption does not alter the convergence properties of the relevant limit since the mistunings are of the same order of magnitude.*

The operator T in our previous discussion was obtained under the assumption that the jitter is observed in the neighborhood of time $t = 0$ with the pulse train extending back in time towards $t = -\infty$. We included in our description of T, X and Y the pulse position deviations regardless of whether a pulse was present or not. The operator T was defined by the matrix

$$T = \begin{bmatrix} \dfrac{\alpha_0}{s_0} & \dfrac{\alpha_1 \beta}{s_0} & \dfrac{\alpha_2 \beta^2}{s_0} & \cdots \\[2ex] 0 & \dfrac{\alpha_1}{s_1} & \dfrac{\alpha_2 \beta}{s_1} & \cdots \\[2ex] 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{2}$$

where $\alpha_n = 1$ if a pulse is present at $t = -n\tau$ and equal to zero otherwise; β is a positive number slightly less than unity ($\beta \approx 1 - (\pi/Q)$); and

$$s_i = \sum_{n=0}^{\infty} \alpha_{n+i} \beta^n. \tag{3}$$

* The question of convergence should not be confused with the question of boundedness of the resulting operator or of the operator T.

When $\mathfrak{A}_n = 0$, all the elements in the nth column of T are zero. As we observed in the periodic case, we can eliminate these columns and the corresponding rows without in any way affecting the results. Physically, this amounts to a consideration of jitter only at those positions where pulses were present in the original pulse train. With these minor changes, we represent T in the following manner.

$$T = \begin{bmatrix} \dfrac{1}{S_0} & \dfrac{\beta^{i_1}}{S_0} & \dfrac{\beta^{i_1+i_2}}{S_0} & \cdots \\ 0 & \dfrac{1}{S_1} & \dfrac{\beta^{i_2}}{S_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{4}$$

where

$$S_0 = 1 + \sum_{n=1}^{\infty} \prod_{\nu=1}^{n} \beta^{i_\nu}, \tag{5}$$

and

$$S_{n-1} = 1 + \beta^{i_n} S_n . \tag{6}$$

Vectors X and Y are also assumed to be suitably modified.

Though we are not concerned with it, we take note of the fact that an alternate formulation of the problem is possible by assuming that the pulse train starts at time $t = 0$ and extends towards $t = +\infty$. There are many disadvantages in such a formulation and we mention it here only for completeness. The operator of interest in this case takes the following form.

$$T_2 = \begin{bmatrix} \dfrac{1}{S_{0+}} & 0 & 0 & \cdots \\ \dfrac{\beta^{i_1}}{S_{1+}} & \dfrac{1}{S_{1+}} & 0 & \cdots \\ \dfrac{\beta^{i_1+i_2}}{S_{2+}} & \dfrac{\beta^{i_2}}{S_{2+}} & \dfrac{1}{S_{2+}} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{7}$$

where

$$S_{0+} = 1, \tag{8}$$

and

$$S_{n+} = 1 + \beta^{i_n} S_{(n-1)+} . \tag{9}$$

Referring back to (1), we are interested in determining whether the mean and the variation about the mean of Y are bounded or not. The averages are to be taken over the components of Y. For our purposes, we shall not be concerned with evaluating any averages. As shown in Part I, the dominant part of $\bar{Y}$, the mean of Y, is the element representing flat delay in the jitter Y. All we need to know is whether the dispersion (or, the spread) about this flat delay remains bounded or not. Though this dispersion has some relation to the variance, it is not the variance. However, we shall continue to use the term variance for the dispersion about the flat delay. The relation between these quantities is shown in Part I. Moreover, the behavior of the dispersion also gives information about the spacing jitter. It also answers the question about the worst pattern.

III. BOUNDEDNESS OF T

We proceed now to examine the operator T to determine some of its important properties. It will be shown here that T is a bounded linear operator which maps the normed linear space $\mathbf{l}_p (1 \leqq p \leqq \infty)$ into itself.*

Theorem: The norm of T (i.e., $| T |$) on $\mathbf{l}_p$ is bounded for each p.†

Proof: Define a diagonal matrix

$$D = \text{diag}\cdot\{S_0^{-1}, S_1^{-1}, S_2^{-1}, \cdots\},$$

and a matrix T_0 such that

$$T = DT_0 .$$

Then,

$$\begin{aligned} | T | &= | DT_0 | \leqq | D || T_0 | \\ &\leqq | T_0 |, \quad (| D | \leqq 1), \\ &= | I + \text{diag}\cdot\{\beta^{i_1}, \beta^{i_2}, \beta^{i_3}, \cdots\}S + \text{diag}\cdot\{\beta^{i_1+i_2}, \beta^{i_2+i_3}, \cdots\}S^2 \\ &\qquad + \cdots | ; \end{aligned}$$

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* The space $\mathbf{l}_p$ is the linear space of all sequences $x = \{\alpha_n\}$ of scalars for which the norm $| x | = \{ \sum_{n=1} | \alpha_n |^p\}^{1/p}$ is finite. The norm for $\mathbf{l}_\infty$ is

$$| x | = \sup_n | \alpha_n | .$$

For precise terminology and definitions as well as a basis for many of the statements made and concepts used in this paper, the reader should consult: N. Dunford and J. T. Schwartz, *Linear Operators — Part I: General Theory*, Interscience Publishers, Inc., New York, N. Y.; 1958.

† The bound or norm of T defined on a linear space χ is the $\sup_{|x| \leqq 1} | Tx |$, denoted by $| T |$. The operator T is bounded if $| T | < \infty$.

here

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is defined on l_p with $| S | = 1$ for each value of p. Note that $0 < \beta < 1$ and $i_\nu \geqq 1$ for $\nu = 1, 2, 3, \cdots$. So

$$\begin{aligned} | T | &\leqq | T_0 | \\ &\leqq \sum_{\nu=0}^{\infty} |(\beta S)^\nu | \\ &\leqq \sum_{\nu=0}^{\infty} | \beta S |^\nu \\ &= \frac{1}{1 - | \beta S |} \end{aligned}$$

since $| \beta S | < 1$. The norm of T is shown to be bounded for each p.

As we shall see in the next section, the space l_∞ is of particular interest to us. The norm of T on l_∞ is given by the supremum of the sum of the absolute values of elements in a row. Since T is a stochastic matrix, $| T | = 1$ when it is defined on l_∞.

IV. DOMAIN OF T

It has been stated before that for our problem the domain of the operator T is the space l_∞. This is not a separable space and, hence, it is not the most convenient one to work with. It must clearly be understood, therefore, that the problem is defined on this space not due to preference but out of necessity. In our discussion of this matter, we begin with some observations in physical terms about the domain in question.

The operator T operates on the sequence representing the jitter at the output of the first repeater (or, the jitter input at the second repeater). The domain of T must include the set of all jitter sequences at the output of the first repeater.* The nature of these sequences is

* Here, we are concerned not with a specific operator but with the totality of the operators.

determined by two essential properties of the original pulse trains, viz., infinite length and random character. Since the pulse trains can be indefinitely long and completely random, the jitter sequences need not all converge to zero or to any other value. This conclusion is valid regardless of whether we consider jitter at all the possible pulse positions or only where the pulses are present. As a consequence of the above conclusion, and since the set of all the jitter sequences is certainly not a finite set, the domain cannot be any of the spaces $\mathbf{l}_p$ with p finite. It also follows that the domain cannot be either c_0 (the space of sequences converging to zero), or c (the space of convergent sequences). These are separable spaces and they are to be preferred over $\mathbf{l}_\infty$ if we are able to represent the problem in terms of any one of them. However, the above discussion shows that this is not possible.

On the other hand, if the jitter sequences are all bounded sequences, then the domain of T can be $\mathbf{l}_\infty$. Obviously, the jitter sequences must be bounded in any realistic situation. In fact, the formulation of the problem assumes that the jitter introduced by a single repeater is quite small compared to 2π radians. Thus, the jitter sequences are all bounded and the domain of T is $\mathbf{l}_\infty$.

A more precise bound on the jitter sequences can be obtained quantitatively. The jitter sequences are defined by

$$\{\beta \dot{S}_n S_n^{-1}\}, \tag{10}$$

where S_n are defined in (6) and $\dot{S}_n = (d/d\beta)S_n$. The bound on any sequence of the above type exists, and it can be obtained by determining the worst case as discussed by Aaron and Gray.* It is also clearly seen from (10) that the sequences need not all necessarily converge to zero (or, to any other value). We see now, in a precise manner, that the domain of T must indeed be $\mathbf{l}_\infty$.

V. SPECTRUM OF T

So far we have established that all the jitter sequences at the input of the second repeater are elements of the space $\mathbf{l}_\infty$, and the operator T is a bounded operator defined on $\mathbf{l}_\infty$ with $|T| = 1$. We recall that the jitter accumulation in a string of repeaters is given in terms of a function of the operator T. In order to determine the properties of a function

* M. R. Aaron and J. R. Gray, Probability Distribution for the Phase Jitter in Self-Timed Reconstructive Repeaters for PCM, B.S.T.J., **41**, March, 1962; pp. 503–558.

of an operator it is necessary to start with some knowledge of the spectrum of the operator.*

The operator T is represented by a triangular matrix. We wish to emphasize that, for an infinite triangular matrix, the diagonal elements are not necessarily the eigenvalues of the matrix. Equally important is the observation that the set of eigenvalues may indeed include elements which are not to be found on the main diagonal.† Moreover, the spectrum of T may also contain points other than those in the point spectrum (i.e., the set of eigenvalues). Therefore, even though T is represented by a triangular matrix it is not a trivial matter to determine its spectrum.

Of course, T is a stochastic matrix and so $\lambda = 1$ is an eigenvalue of T with the corresponding eigenvector $x_0 = \{1, 1, 1, \cdots\}$. Some other results also follow from the stochastic nature of T. We shall denote the spectrum of T by $\sigma(T)$.

Theorem: The spectrum of T is a subset of the unit disk (i.e., $|\sigma(T)| \leqq 1$), and any pole λ of T with $|\lambda| = 1$ has order one.‡

Proof: The first statement follows immediately from the fact that $|T| = 1$. It is well known that for any λ such that $|\lambda| > |T|$ the resolvent operator $(\lambda I - T)^{-1}$ exists. Thus, the spectral radius of T, viz., $\sup |\sigma(T)|$ cannot exceed one. The spectrum is a subset of the unit disk, etc.

In order to prove the second statement, it suffices to treat the case that $\lambda = 1$ is a pole of T. Or else we treat a modified operator (T/λ) with norm one for $|\lambda| = 1$. Suppose that the order of the pole is at least two. Then there must exist an $x_0 \in E(1; T)\chi$, such that $(I - T)x_0 \neq 0$, but $(I - T)^2 x_0 = 0$.§ Consider a function of T corresponding to $f(\lambda) = \lambda^n/n$ in the neighborhood of $\lambda = 1$. We obtain a relation of the form

$$\frac{1}{n} T^n x_0 = \frac{1}{n} x_0 + (I - T)x_0 .$$

Letting $n \to \infty$, we conclude that $(I - T)x_0 = 0$, which is a contradiction. Hence the poles of T which lie on the unit circle are simple poles.

* The spectrum $\sigma(T)$ of T is the complement of $\rho(T)$. The resolvent set $\rho(T)$ of T is the set of scalars λ, for which $(\lambda I - T)^{-1}$ exists as a bounded operator with domain χ, where χ is the domain of T. The function $R(\lambda; T) = (\lambda I - T)^{-1}$, defined on $\rho(T)$, is the resolvent of T.

† We hope to discuss elsewhere these statements and their implications in greater detail and with reference to linear operators in general.

‡ An isolated point λ_0 of $\sigma(T)$ is called a pole of T if $R(\lambda; T)$ has a pole at λ_0. By the order $\nu(\lambda_0)$ of a pole λ_0 is meant the order of λ_0 as a pole of $R(\lambda; T)$.

§ $E(\lambda_0; T)$ is a function of T which is identically one on a pole λ_0 of T but which vanishes on the rest of $\sigma(T)$. Observe that E is a projection operator, i.e., $E^2 = E$. The definition of E given here is a highly restricted one but it suits our purposes.

The next two theorems give us some more information about the spectrum. The first one shows that there cannot be a pole on the unit circle for $\lambda \neq 1$. The second one concerns the dimension of the eigenmanifold corresponding to the eigenvalue $\lambda = 1$.*

Theorem: All points on the unit circle except $\lambda = 1$ *are in* $\rho(T)$.

Proof: We already know that $\lambda = 1$ is in $\sigma(T)$. We also know that any λ such that $|\lambda| > 1$ is in $\rho(T)$. To show that any $\lambda \neq 1$ on the unit circle is in $\rho(T)$, consider

$$R(\lambda_0 ; T) = (\lambda_0 I - T)^{-1}, \qquad \{\lambda_0 \neq 1, \qquad |\lambda_0| = 1\}.$$

If we can show that $R(\lambda_0 ; T)$ exists for all x in χ with a bounded norm, we have proved the theorem. It is easy to verify that $R(\lambda_0 ; T)$ may be expressed as shown in (11).

$$R(\lambda_0 ; T) =$$

$$\begin{bmatrix} \dfrac{1}{(\lambda_0 - S_0^{-1})} & \dfrac{\beta^{i_1} S_0^{-1}}{(\lambda_0 - S_0^{-1})(\lambda_0 - S_1^{-1})} & \dfrac{\beta^{j_1+i_2} S_0^{-1} \lambda_0}{(\lambda_0 - S_0^{-1})(\lambda_0 - S_1^{-1})(\lambda_0 - S_2^{-1})} & \cdots \\ 0 & \dfrac{1}{(\lambda_0 - S_1^{-1})} & \dfrac{\beta^{i_2} S_1^{-1}}{(\lambda_0 - S_1^{-1})(\lambda_0 - S_2^{-1})} & \cdots \\ 0 & 0 & \dfrac{1}{(\lambda_0 - S_2^{-1})} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \qquad (11)$$

Since λ_0 is a complex number, it follows that $(\lambda_0 - S_i^{-1}) \neq 0$ for any i. Next, we show that $R(\lambda_0 ; T)$ is a bounded operator. Observe that the norm is given by

$$|R(\lambda_0 ; T)| = \sup_i \sum_j |a_{ij}|, \qquad (12)$$

where, a_{ij} represents the element in the ith row and jth column of the matrix in (11), i.e., $R(\lambda_0 ; T) = \| a_{ij} \|$.

Consider the resolvent $R(\lambda_1 ; T)$ for $\lambda_1 = (1 + \epsilon)$ with $\epsilon > 0$. Obviously λ_1 is in $\rho(T)$ and $|R(\lambda_1 ; T)| < \infty$. We assert that, given any λ_0, there exists an $\epsilon > 0$ such that

$$|R(\lambda_0 ; T)| \leqq |R(\lambda_1 ; T)| < \infty. \qquad (13)$$

The validity of our assertion is proven by first noting that $R(\lambda_1 ; T)$ is represented by the matrix in (11) with λ_0 replaced by λ_1. Let $R(\lambda_1 ; T) = \| b_{ik} \|$. Next we show that $|a_{ik}| \leqq |b_{ik}|$, for all i and k, from which follows relation (13). Let $\lambda_0 = \cos\theta + j \sin\theta$, $(j = \sqrt{-1})$. Then

* If $\lambda = 1$ is a pole of T, this is the dimension of the range of projection $E(1; T)$.

$$\left|\frac{a_{mn}}{b_{mn}}\right| = \left|\frac{\lambda_0}{\lambda_1}\right|^{(n-m-1)} \prod_{\nu=m}^{n} \left|\frac{\lambda_1 - S_\nu^{-1}}{\lambda_0 - S_\nu^{-1}}\right|, \qquad n \geqq (m+1),$$

$$= \left|\frac{\lambda_1 - S_m^{-1}}{\lambda_0 - S_m^{-1}}\right|, \qquad n = m,$$

for $m = 0, 1, 2, \cdots$. In any case, for $n \geqq m$,

$$\left|\frac{a_{mn}}{b_{mn}}\right| \leqq \prod_{\nu=m}^{n} \left|\frac{\lambda_1 - S_\nu^{-1}}{\lambda_0 - S_\nu^{-1}}\right| \quad \text{since} \quad |\lambda_0| < |\lambda_1|.$$

Consider a term of the form

$$\left|\frac{\lambda_1 - \alpha}{\lambda_0 - \alpha}\right|,$$

where $\alpha = (1 - \beta)$ is the lower bound on S_ν^{-1}. Then

$$\left|\frac{\lambda_1 - \alpha}{\lambda_0 - \alpha}\right| = \left|\frac{1 + \epsilon - \alpha}{\cos\theta - \alpha + j\sin\theta}\right|$$

$$= \left[\frac{(1 + \epsilon - \alpha)^2}{1 + \alpha^2 - 2\alpha\cos\theta}\right]^{\frac{1}{2}}$$

$$\leqq 1$$

provided that

$$\epsilon^2 + 2\epsilon(1 - \alpha) - 2\alpha(1 - \cos\theta) \leqq 0.$$

Since $0 < \alpha < 1$, the polynomial on the left side has one zero for $\epsilon > 0$ and one zero for $\epsilon < 0$. There exists, therefore, an $\epsilon > 0$ such that the above inequality is satisfied as long as $\theta \neq 0$. Since

$$\left|\frac{\lambda_1 - S_\nu^{-1}}{\lambda_0 - S_\nu^{-1}}\right| \leqq \left|\frac{\lambda_1 - \alpha}{\lambda_0 - \alpha}\right|, \qquad \alpha \leqq S_\nu^{-1} \leqq 1,$$

$$\leqq 1,$$

it follows that

$$\left|\frac{a_{mn}}{b_{mn}}\right| \leqq \prod_{\nu=m}^{n} \left|\frac{\lambda_1 - S_\nu^{-1}}{\lambda_0 - S_\nu^{-1}}\right|, \qquad n \geqq m$$

$$\leqq \left|\frac{\lambda_1 - \alpha}{\lambda_0 - \alpha}\right|^{n-m+1}$$

$$\leqq 1.$$

$$|a_{mn}| = |b_{mn}| = 0, \qquad \text{for } n < m.$$

The theorem is thus proved, and all the points on the unit circle except $\lambda = 1$ are in the resolvent set $\rho(T)$.

It follows from the above theorem that there are no poles on the unit circle except possibly at $\lambda = 1$. We know that such a pole, if it exists, must be of order one. The next theorem concerns the dimension of the eigenmanifold corresponding to $\lambda = 1$.

Theorem: There exists one and only one nontrivial element $x \in \chi$ such that $Tx = x$.

Proof: It is apparent that $x_0 = \{1, 1, 1, \cdots\}$ is one such element. If there exists another element $x \neq x_0$ (but, $|x| = |x_0|$), then some of its components must be unequal. Let $x = \{\xi_0, \xi_1, \xi_2, \cdots\}$. Then there is some $\xi_n \neq \xi_{n+1}$. We will show that this is impossible.

If $Tx = x$, it follows that [cf. (4)]

$$\xi_n = S_n^{-1}\xi_n + S_n^{-1}\beta^{i_{n+1}}\xi_{n+1} + S_n^{-1}\beta^{(i_{n+1}+i_{n+2})}\xi_{n+2} + \cdots$$

and

$$\xi_{n+1} = S_{n+1}^{-1}\xi_{n+1} + S_{n+1}^{-1}\beta^{i_{n+2}}\xi_{n+2} + \cdots .$$

Substituting the second equation into the first we obtain

$$\xi_n = S_n^{-1}\xi_n + S_n^{-1}\beta^{i_{n+1}}S_{n+1}\xi_{n+1} .$$

Or, since from (6)

$$S_n - 1 = \beta^{i_{n+1}}S_{n+1} ,$$

we have a contradiction

$$\xi_n = \xi_{n+1} .$$

This proves the theorem, and the eigenmanifold corresponding to $\lambda = 1$ is of dimension one.

The results obtained in this section about the spectrum of T are quite general and remain valid under any restriction of the domain $\mathbf{l}_\infty$ assuming, of course, that x_0 is in such a restriction. The all-important question not answered in this section is whether or not T has a pole at $\lambda = 1$. This is a crucial question indeed and, on the basis of the results already obtained, the answer determines the behavior of the variance of the jitter. We delay the discussion of the existence of a pole at $\lambda = 1$ in order to first show its pivotal character. Next, we show that the existence of the pole depends upon a certain suitable restriction of the domain of T. These two points lead us to our final conclusions.

VI. BOUNDEDNESS OF VARIANCE

Let us consider now what happens to the output jitter as the number of repeaters approaches infinity. We obtain the results, at first, under the assumption that T has a pole at $\lambda = 1$. We discuss later the case where $\lambda = 1$ is not a pole of T.

Theorem: If $\lambda = 1$ is a pole of T, then there exists a bound on the variance of

$$y = [\sum_{m=0}^{\infty} T^m]x. \tag{14}$$

Proof: Let $\lambda = 1$ he a pole of T. Then $\sigma(T)$ may be decomposed into the union of a closed set σ, which lies inside a circle $| Z | < \alpha_0 < 1$, and the simple pole at $\lambda = 1$. Let us put $E_1 = E(1; T)$, $E_D = (I - E_1)$ and $D = TE_D$.* The range of E_1 is one-dimensional, and the iterates of T are given by

$$T^m = E_1 + D^m, \tag{15}$$

since for a simple pole at $\lambda = 1$

$$f(T)E_1 = f(1)E_1,$$

and

$$T^m E_D = D^m.$$

It also follows that $\sigma(D) = \sigma + \{0\}$, and so $\sigma(D)$ is contained in the disk $| Z | < \alpha_0$ for some $\alpha_0 < 1$. From the definition of spectral radius, this implies that $\limsup_{m\to\infty} | D^m |^{1/m} < \alpha_0$, from which it follows that for $m \geqq 1$,

$$| D^m | \leqq M\alpha_0^m \tag{16}$$

for some positive number M.

Next, observe that the space χ is a direct sum of subspaces $\chi_1 = E_1\chi$ and $\chi_D = E_D\chi$, which are invariant under T since T commutes with E_1 and E_D. It follows from (15) and (16) that

(a) $Tx = x$, for x in χ_1;

(h) $T^n x \to 0$ exponentially fast, for x in χ_D.

Every x in (14), then, is given by

$$x = x_1 + x_D,$$

where $x_1 = E_1 x$ and $x_D = E_D x$. The element x_1 except for a constant

* Observe that E_D is also a projection operator since $E_D{}^2 = E_D$.

multiplier is the eigenvector $\{1, 1, 1, \cdots\}$. Then

$$y = \lim_{n \to \infty} [\sum_{m=0}^{n} T^m](x_1 + x_D).$$

Obviously, the mean of y increases indefinitely* since

$$\bar{y} = \lim_{n \to \infty} [\sum_{m=0}^{n} T^m]x_1 + \bar{y}_D , \tag{17}$$

where

$$y_D = \lim_{n \to \infty} [\sum_{m=0}^{n} T^m]x_D . \tag{18}$$

The first term on the right-hand side of (17) increases indefinitely, and so $\bar{y} \to \infty$. The limit in (18) exists [cf. (16) and statement (b) above] and so

$$[y - \bar{y}] = y_D - \bar{y}_D \tag{19}$$

is bounded. Hence, the variance is bounded, if $\lambda = 1$ is a pole of T, as was to be proved. The physical interpretations of this case are discussed in the concluding section.

It must be observed that the bound on the variance is shown to exist for all elements x in χ. Hence, the result is valid for the admissible elements, viz., the jitter sequences.

The boundedness of the variance is a consequence of the inequality (16). As a function of α_0, the bound varies as $(1 - \alpha_0)^{-1}$ and increases indefinitely as α_0 approaches one. Therefore, we ask whether infinity is, indeed, the least upper bound on the variance when $\lambda = 1$ is *not* a pole of T. We anticipate the results of the next section to state that there is no bound (finite) on the variance when 1 is not a pole of T. We first show that given any number M, there exists an element x in χ, such that the variance of y exceeds M. Next, we show that there exist admissible elements for which the same conclusion holds.

VII. UNBOUNDED VARIANCE

We show, at first, that $\lambda = 1$ is not a pole of T in the general case. By the general case, we mean that the domain is not restricted in any way.

Theorem: The point $\lambda = 1$ is the limit point of the point spectrum of T.

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* As discussed in Part I, there exists at least one X such that $x_1 \neq 0$.

Proof: We first determine the conditions that x must satisfy for $Tx = \lambda x$. Let $x = \{\xi_0, \xi_1, \cdots\}$. Then, if $Tx = \lambda x$,

$$\lambda\xi_n = S_n^{-1}\xi_n + S_n^{-1}\beta^{i_{n+1}}\xi_{n+1} + S_n^{-1}\beta^{(i_{n+1}+i_{n+2})}\xi_{n+2} + \cdots$$

and

$$\lambda\xi_{n+1} = S_{n+1}^{-1}\xi_{n+1} + S_{n+1}\beta^{i_{n+2}}\xi_{n+2} + \cdots.$$

Substituting the second equation into the first, we obtain

$$(\lambda - S_n^{-1})\xi_n = \lambda S_n^{-1}S_{n+1}\beta^{i_{n+1}}\xi_{n+1}.$$

Or, since

$$S_n - 1 = \beta^{i_{n+1}}S_{n+1},$$

we have

$$\xi_{n+1} = \frac{S_n - (1/\lambda)}{S_n - 1}\,\xi_n, \qquad (n = 0, 1, 2, \cdots). \tag{20}$$

From (20) we note that when $\lambda = S_n^{-1}$ we obtain an eigenvector x with $(n + 1)$ nonzero elements $\xi_k(k = 0, 1, \cdots, n)$. Hence, if the diagonal elements S_n^{-1} approach one as $n \to \infty$, then $\lambda = 1$ is a limit point of the set of eigenvalues. However, of greater physical importance is the case when the number of successive zeros in the admissible pulse trains has a finite upper bound. In such cases, the diagonal elements have an upper bound less than unity, i.e.,

$$S_n^{-1} \leqq \alpha < 1. \tag{21}$$

Even in these cases, there exists an eigenvector x for every λ such that $\alpha < \lambda \leqq 1$. We obtain the vector x from (20), starting with $\xi_0 = 1$. Since $S_n^{-1} < \lambda \leqq 1$, we find that the sequence $\{\xi_n\}$ is a strictly decreasing sequence, i.e.,

$$0 \leqq \xi_{n+1} < \xi_n \neq 0.$$

The sequence $x = \{\xi_n\}$ converges to zero, and hence it is a member of the space c_0 and has norm one. A simple substitution of x, obtained from (20), into the equation $Tx = \lambda x$ shows that x is indeed an eigenvector. Since an eigenvector x exists for every λ such that $\alpha < \lambda \leqq 1$, the point $\lambda = 1$ is the limit point of the point spectrum of T. The proof is complete and $\lambda = 1$ is not a pole of T.

It immediately follows that when all x in χ are admissible elements, there exists no bound on the variation of y about the flat delay. If it does, let M be such a bound. Then we can always find an eigenvector x,

corresponding to a $\lambda_1 > \alpha$, such that $(1 - \lambda_1)^{-1} > M$. Since x_1 is a member of c_0, the flat delay in the jitter is zero. The dispersion is given by

$$y_1 = \left(\frac{1}{1 - \lambda_1}\right) x_1 ;$$

and

$$| y_1 | > M | x_1 |,$$

which is a contradiction. Hence, there is no bound, etc.

To show that the same conclusion holds when the admissible elements x are the jitter sequences, we need merely show that there exists an admissible jitter element x in c_0 such that $(x - x_1)$ is nonnegative, i.e., nonnegative elements in the sequence $(x - x_1)$. Then, since all elements of T are nonnegative, $| Tx | \geqq | Tx_1 | > M | x_1 |$. Such an element x can be constructed easily by letting all pulses be present for a long enough time and then letting one of the pulses be absent, after which there is a string of alternating pulse and space, and then two pulses are absent, and so on. The sequence x for this case is a member of c_0 since the jitter will ultimately approach zero. The elements of x are assuredly greater than those of x_1 provided we make the string of pulses long enough between spaces.*

Similar conclusions are valid when the number of successive zeros in the original pulse train does not exceed a specified finite number. In this case, we use a member of the space c, $x = x_0 + x_1$, where x_1 is defined above and x_0 is the eigenvector corresponding to $\lambda = 1$. The dispersion is, as before,

$$y_1 = \left(\frac{1}{1 - \lambda_1}\right) x_1 .$$

The admissible jitter sequence is one that converges to x_0 but otherwise has properties similar to the previous case. Physically, the pulse train converges to a periodic pulse train with one pulse and at most the maximum number of successive zeros in each period.

We have thus shown that the bound on the variation of the jitter about the mean exists if T has a pole at $\lambda = 1$ and that there exists no such bound otherwise. At this point, we recall that a somewhat different formulation of the problem is obtained in (7). Let us note here that in the alternate formulation somewhat different but similar

* In fact, numerous admissible sequences with the same properties can be easily constructed. Their linear combination would also be such a sequence, and so on.

development takes place. In the alternate formulation, the point $\lambda = 1$ is not a limit point of the point spectrum. But, in general, neither is it a pole of T. It can be shown that every neighborhood of $\lambda = 1$ contains points in the spectrum of T. From this fact, the rest of the conclusions follow.

VIII. DISCUSSION OF RESULTS

The results may be stated simply in terms of the existence of a pole of T at $\lambda = 1$. If $\lambda = 1$ is in tbe point spectrum of T and it is an isolated point of the spectrum of T (i.e., it is a pole), then the variance of the jitter is bounded. Otherwise, the jitter dispersion has no bound. We show that, in the random case, $\lambda = 1$ is not a pole of T. The same result is obtained when a constraint is put on the number of successive zeros in the pulse train. Thus, there exists no bound on the variation of the jitter about its mean value for the truly infinite and random pulse trains.

On the other hand, of some physical importance are the cases which may be approximated by periodic pulse trains or nonperiodic pulse trains which either are finite or become periodic after a finite interval.* For such cases, the operator T is restricted to a finite dimensional space and $\lambda = 1$ is necessarily a pole of T. The variance is, therefore, bounded. Of course, the bound is a function of the dimension of the space as well as of the other eigenvalues in the spectrum. Each case must be investigated separately to determine the corresponding bound. Such a bound may be all tbat is important in the usual situation where a finite chain of repeaters is present in the system. Some practical means of determining the bounds will be discussed in a subsequent paper. We shall also discuss there many other practical matters, such as errors involved in our model, transients, etc.

To sum up, as the number of repeaters gets larger, the dimension of the space gets larger (since tbe effective pulse train gets longer), and the maximum dispersion of the jitter increases. Thus, there is such a thing as a worst pattern when there are a finite number of repeaters. However, the worst value of the jitter keeps on increasing.

The rate at which the variance increases as a function of the number of repeaters is not investigated in this paper. It is, of course, not possible for the dispersion to grow faster than n, the number of repeaters. This conclusion follows from the fact that the norm of T is equal to one.

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* Many other physical constraints may be used to restrict T to a finite dimensional space. The variance is bounded in all such cases.

More precise determination of the rate of growth would depend upon a particular distribution of the random variables involved. We do not pursue this aspect of the problem.*

The conclusions about the spacing jitter (cf. Part I) follow along the same lines as above for the finite and infinite dimensional spaces. The misalignment, $T^n x$, in the nth repeater is also influenced by the dimensionality of the domain of T. When the dimension is finite, the misalignment is merely a flat delay (since $\lambda = 1$ is an isolated eigenvalue) for reasonably large n. However, when there is no pole at $\lambda = 1$, the misalignment is not so simply stated, but it is different from repeater to repeater.

We conclude with the observation that the approach proposed here should be potentially useful for many problems of signal processing encountered in data systems.

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* For some partial results, for a somewhat different model, refer to C. J. Byrne, B. J. Karafin and D. B. Robinson, Jr., Pattern Induced Timing Jitter in T-1 PCM Repeaters, to be published. This paper uses a model proposed, in an unpublished report, by R. C. Chapman, Jr.